Journal of Statistical Physics, Vol. 29, No. 4, 1982

Clustering in Low-Dimensional $SO(N)$-Invariant Statistical Models with Long-Range Interactions

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Received May 25, 1982

We obtain upper bounds for the two-point correlation functions in statistical models in one or two dimensions which have $SO(N)$ symmetry. This clarifies upper bounds for long range interactions for which there exists clustering.

KEY WORDS: Clustering; long-range potential; lattice Green's function; reflection positivity.

1. INTRODUCTION AND MAIN THEOREMS

There has been some interest in statistical models in low dimensions which have long-range interactions.[1-3] In this paper we discuss (translationally invariant) statistical models which have $SO(N)$-invariant two-body interactions. As is quite well known,[10] there exists no spontaneous magnetization in these models if the dimension is less than or equal to 2. This is not the case for long-range interactions or for long-range potentials.

This kind of model typically has the following Hamiltonian:

$$H_\Lambda(s) = \sum_{(x,y)\in\Lambda\otimes\Lambda} j(x-y)s_x s_y \tag{1}$$

where $s_x \in S^{N-1}$ ($N \geqslant 2$), Λ a bounded (rectangular) region in Z^ν, $\{j(n) = j(-n) \in R;\ n \in Z^\nu\}$ is the potential, and a pair (x, y) is taken only once in the sum. Furthermore, we assume

$$\tfrac{1}{2}\sum |j(n)| = 1, \qquad j(0) = 0$$

Supported by the SRC.

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0022-4715/82/1200-0747$03.00/0

The corresponding Gibbs measure $d\mu_\Lambda$ is given by

$$\frac{1}{Z_\Lambda}\exp\left[\ \beta H_\Lambda(s)\right]\prod_x ds_x \tag{2}$$

where Z_Λ is the normalization constant chosen so that $\int d\mu_\Lambda = 1$.

We also discuss quantum models. However, in this paper we restrict ourselves to the Heisenberg model without loss of generality, where we replace s_x by $\sigma_x = (\sigma_x^1, \sigma_x^2, \sigma_x^3)$ and $\int(\cdot)\,d\mu_\Lambda$ by $\mathrm{Tr}[(\cdot)\exp\beta H_\Lambda]/Z_\Lambda$, respectively. Here $(\sigma_x^i)^* = \sigma_x^i \in \mathrm{Mat}((2S+1)\times(2S+1))$, and

$$\left[\sigma_x^i, \sigma_x^j\right] = i\epsilon_{ijk}\sigma_x^k, \qquad \sum_{i=1}^{3}\left(\sigma_x^i\right)^2 = S(S+1) \tag{3}$$

For given $\{j(n) = j(-n);\ n \in Z^\nu\}$, we consider a positive potential $\{J(n) = J(-n) \geqslant 0;\ n \in Z^\nu\}$ such that

$$|j(n)| \leqslant J(n), \qquad J(0) = 0, \qquad \sum J(n) < \infty \tag{4}$$

We define a generalized lattice Laplacian Δ^J by

$$(-\Delta^J)(x,y) = \begin{cases} \sum_n J(n), & x = y \\ -J(n), & x = y + n \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Let

$$\tilde{C}(k) = \left[\sum_{x\in Z}(-\Delta^J)(x,0)e^{ikx}\right]^{-1} = \left[\sum_n J(n)(1-\cos kn)\right]^{-1} \tag{6}$$

be the Fourier transform of the Green's function $(-\Delta^J)^{-1}(x,0)$, where $k \in [-\pi,\pi)^\nu$ and we have used $J(n) = J(-n)$. Obviously $\tilde{C}(k)^{-1} \geqslant 0$. Since $\sum J(n) < \infty$ has been assumed, $\tilde{C}(k)^{-1}$ is continuous and $\tilde{C}(k)^{-1} \to 0$ as $|k| \to 0$. Since $(\Delta^J)^{-1}$ may not be defined for $\nu = 1$ or 2, we define a lattice Green's function C_0 by

$$C_0(x) = \int \frac{d^\nu k}{(2\pi)^\nu}\left(e^{ikx} - 1\right)\tilde{C}(k) \tag{7}$$

Then $C_0(x) \leqslant C_0(0) = 0$. Let us assume that $\tilde{C}(k)$ is not integrable in a neighborhood of $k = 0$. Then

$$\lim_{|x|\to\infty} C_0(x) = -\infty \tag{8}$$

and what we claim is that there exists clustering whenever $\tilde{C}(k)$ is not integrable in a neighborhood of $k = 0$.

We may assume that $\{J(n)\}$ satisfies

(A) $\nu = 1$: $\{J(n)\}$ is such that $C_0(x) = C_0(|x|)$ is monotone decreasing and convex in $|x|$.

$\nu = 2$: $\{J(n)\}$ is such that $C_0(x)$ is (approximately) rotationally invariant, and monotone decreasing and convex in $|x| = (x_1^2 + x_2^2)^{1/2}$. Namely, $C_0(x) = C(|x|) + O(1)$, $C(|x|)$ = monotone decreasing and convex.

For (A) to hold, the following (A′) is sufficient (see Appendix):

(A′) $\nu = 1$: $\{J(n) = J(-n) \geqslant 0\}$ is reflection positive (RP).[3]

$\nu = 2$: $\{J(n) = J(-n) \geqslant 0\}$ is (approximately) rotationally invariant and reflection positive for the x_1 direction. Furthermore: (1) $J(n) \leqslant c_1|n|^{-3-\epsilon_1}$, (2) $|J(m) - J(n)| \leqslant c_2|n|^{-4-\epsilon_2}$ for $||m| - |n|| \leqslant 1$, and (3) $J(n) = J((|n_1|, |n_2|) = J(|n_2|, |n_1|))$ for any $n = (n_1, n_2) \in Z^2$, where c_i and ϵ_i are positive constants.

There are sufficiently many $J(n)$'s which satisfy (A) or (A′). For example, the following are RP (for $\nu = 1, 2, \ldots$):

(i) $J(n) = \text{const}\, n^{-\alpha}$, $\alpha > 0$,
(ii) $J(n) = \text{const}(1/n^{\alpha})\log^{\alpha_1} n \cdots \log_s^{\alpha_s} n$, $\alpha > 0$,

where $n = |n|$ and $\log_s n = \log \cdots \log n$ (s times). See Appendix or Refs. 3, 6, 7 for the details.

These assumptions may be used in the classical or quantum $SO(N)$-invariant models, but are not necessary at all in the case of the Villain model, which is a variation of the XY model and has the $U(1)$ symmetry. And moreover we can obtain an almost optimal upper bound for this model. Then it is quite meaningful to consider the Villain model, even though this is defined only for positive potentials. Let

$$\mathcal{V}_{\beta J(n)}(\theta) = \sum_{l=-\infty}^{\infty} \exp\left[-\frac{\beta}{2} J(n)(\theta + 2l\pi)^2 \right] \tag{9}$$

This is almost equal to $\text{const}\exp[\beta J(n)\cos\theta]$ for large $\beta J(n)$. We define the Gibbs measure of the Villain model by

$$\frac{1}{Z} \prod_{(x,y)} \mathcal{V}_{\beta J(x-y)}(\theta_x - \theta_y) \prod \frac{d\theta_x}{2\pi} \tag{10}$$

where $\{J(n)\}$ has been used instead of $\{j(n)\}$ since the potential must be positive in this model.

Theorem 1. In the Villain model,

$$\langle s_0 s_\zeta \rangle = \langle \cos(\theta_0 - \theta_\zeta) \rangle \leqslant \exp\left[\frac{1}{\beta} C_0(\zeta) \right] \tag{11}$$

for any potential $\{J(n)\}$.

Theorem 2. Assume that there exists a positive potential $\{J(n)\}$ which satisfies inequality (4) and condition (A) or (A′). Then for both classical and quantum models,

$$|\langle s_0 s_\zeta \rangle| \leqslant \text{const}\left[-C_0(\zeta) \right]^{-\gamma}, \qquad \gamma \leqslant 1 \tag{12}$$

Remarks 1. (1) Our upper bounds are optimal in the sense that they cluster whenever $\tilde{C}(k)$ is not integrable in a neighborhood of $k = 0$. If $\tilde{C}(k)$ is integrable, there exists spontaneous magnetization for large β (if the potentials are RP[3]).

(2) Our upper bound (12) is not optimal because it is known[2,6–8] that for $\nu = 1$ the two-point correlation functions satisfy L^1 clustering provided that $\sum |nj(n)| < \infty$. On the other hand (11) satisfies this requirement as is proved in Lemma A.4. Then we conjecture that (12) may be improved in the form of inequality (11). See Remark 2.

(3) Let $\nu = 1$ for simplicity and let Θ be the reflection operator with respect to $x = 1/2$ (see Ref. 3). Then

$$\begin{aligned} \mathcal{V}_\beta(\theta_0 - \theta_1) &= K\left[1 + 2\sum_1^\infty \exp(-n^2/2\beta) \cos n(\theta_0 - \theta_1) \right] \\ &= K\left\{ 1 + 2\sum_1^\infty \exp(-n^2/2\beta) \right. \\ &\qquad \left. \times \left[\cos n\theta_0 \Theta(\cos n\theta_0) + (\cos \to \sin) \right] \right\} \end{aligned}$$

where $K > 0$. Then the Gibbs measure of the Villain model with nearest-neighbor interaction is RP. For long-range potentials, it seems to be an open problem when the Gibbs measure is RP.

2. PROOF OF THEOREMS

Our proof is an extension of the method employed by McBryan and Spencer[9] in order to obtain upper bounds for the two-point correlation functions in the XY models (with short-range interactions). This method may be called an estimate of integrals by means of complex translations. Even though their method is very powerful for short-range potentials, some tricks are necessary when long-range potentials appear. This is not the case for the Villain model.

Proof of Theorem 1. We first introduce a periodic box $\Lambda = [-N, N)^\nu \cap Z^\nu$ and define the Gibbs measure $d\mu_\Lambda$:

$$\frac{1}{Z_\Lambda} \prod_{x,y \in \Lambda} \mathcal{V}_{\beta \hat{J}(y-x)}(\theta_x - \theta_y) \prod_x \frac{d\theta_x}{2\pi}$$

where the long-range potential $\{J(n) = J(-n) \geqslant 0\}$ is changed into a periodic potential $\{\hat{J}(n) = \hat{J}(-n) \geqslant 0\}$: $\hat{J}(m) = J(n)$ if $m_i = n_i \in [-N, N)$ mod $2N$, and furthermore $\theta_x = \theta_{x'}$ if $x_i = x_i'$ mod $2N$.

Let $a(x) \in R$, and we change the contour $0 \to 2\pi$ of $d\theta_x$ as $0 \to ia(x) \to 2\pi + ia(x) \to 2\pi$. $\mathcal{V}_\beta(\theta)$ is holomorphic in θ and $\mathcal{V}_\beta(\theta + 2n\pi) = \mathcal{V}_\beta(\theta)$ for any $\theta \in C$. Then this change of the contour is equivalent to $\theta_x \to \theta_x + ia(x)$, $\theta_x \in [0, 2\pi)$. Now

$$|\mathcal{V}_\beta(\theta + ia)| \leqslant \mathcal{V}_\beta(\theta) \exp\left(\frac{\beta}{2} a^2\right) \tag{13}$$

Then by using the definition of Z_Λ, we have

$$\begin{aligned} \langle \exp[i(\theta_0 - \theta_\zeta)] \rangle &\leqslant \exp\Big\{ -[a(0) - a(\zeta)] \\ &\qquad + \sum_{x,n} \frac{\beta \hat{J}(n)}{2} [a(x) - a(x+n)]^2 \Big\} \\ &= \exp\left\{ -[a(0) - a(\zeta)] + \frac{\beta}{2} (a, (-\Delta^{\hat{J}}) a) \right\} \end{aligned} \tag{14}$$

where $\Delta^{\hat{J}}$ is the generalized lattice Laplacian defined by Eq. (4). (J is replaced by $\hat{J}$ and periodic boundary conditions are imposed.) Replace $(-\Delta^{\hat{J}})$ by the strictly positive $(-\Delta^{\hat{J}} + M^2) \geqslant (-\Delta^{\hat{J}})$, where $M^2 > 0$. Set

$$\begin{aligned} a(x) &= \frac{1}{\beta} \left[(-\Delta^{\hat{J}} + M^2)^{-1} f \right](x) \\ f(\cdot) &= \delta_{\cdot,0} - \delta_{\cdot,\zeta} \end{aligned} \tag{15}$$

Then we have

$$\langle \ \rangle \leqslant \exp\left[-\frac{1}{2\beta} \left(f, (-\Delta^{\hat{J}} + M^2)^{-1} f \right) \right]$$

Let $M \to 0$ after taking the thermodynamic limit $N \to \infty$. Then

$$\langle \ \rangle \leqslant \exp\left[-\frac{1}{2\beta} \left(f, (-\Delta^J)^{-1} f \right) \right] = \exp\left[\frac{1}{\beta} C_0(\zeta) \right] \quad \blacksquare \tag{16}$$

Proof of Theorem 2 (Classical Spin Cases). As in Ref. 9, it suffices to consider the XY model case ($N = 2$), where the Hamiltonian is given by

$$H = \sum_{x,y} j(y - x)\cos(\theta_x - \theta_y)$$

Now $\cos(\theta + ia) = \cos\theta - i\sin\theta\sinh a + \cos\theta[\cosh a - 1]$. Then using $|\exp \beta j\cos(\theta + ia)| \leqslant \exp[\beta j\cos\theta + \beta|j|(\cosh a - 1)]$ and the definition of Z, we can start with

$$\left|\left\langle \exp\left[i(\theta_0 - \theta_\zeta)\right]\right\rangle\right| \leqslant \exp\left\{-\left[a(0) - a(\zeta)\right] + \beta\sum_{x,n} J(n)\left[\cosh(a(x) - a(x+n)) - 1\right]\right\} \tag{17}$$

Set $K = 2|C_0(\zeta)|$, and without loss of generality we consider the case such that $K \to \infty$ as $|\zeta| \to \infty$. In fact if K is bounded uniformly in ζ, our upper bound is trivial. Then we can also assume $K \geqslant 1$ without loss of generality. Note that

$$\begin{aligned} M(\zeta) &\equiv M(\zeta; J) \\ &\equiv \sup_{x \in Z^\nu, n \in \operatorname{supp} J} |C_0(x) - C_0(x - \zeta) - C_0(x+n) + C_0(x+n-\zeta)| \\ &\leqslant 2\sup_{x \in Z^\nu} |C_0(x) - C_0(x-\zeta)| \end{aligned} \tag{18}$$

By easy geometrical considerations using assumption (A),

$$\sup_{x \in Z^\nu} |C_0(x) - C_0(x - \zeta)| = \begin{cases} |C_0(\zeta)|, & \nu = 1 \\ |C_0(\zeta)| + O(1), & \nu = 2 \end{cases} \tag{19}$$

Namely, the maximum of $|C_0(x) - C_0(x - \zeta)|$ is attained by $x = 0$ or by $x = \zeta$. We choose $a(x)$ as

$$a(x) = \frac{\alpha}{\beta}\frac{\log K}{K}\left[C_0(x) - C_0(x - \zeta)\right] \tag{20}$$

Then

$$|a(x) - a(x+n)| \leqslant (1 + \delta)\frac{\alpha}{\beta}\log K \equiv A \tag{21}$$

where $\delta = 0$ for $\nu = 1$ and $\delta = O(K^{-1})$ for $\nu = 2$. By using the monotone increase of $(\cosh x - 1)/x^2$, we have

$$\cosh\left[a(x) - a(x+n)\right] - 1 \leqslant (1 + \epsilon)A^{-2}e^{A}\tfrac{1}{2}\left[a(x) - a(x+n)\right]^2$$

where $\epsilon \leqslant 1$ and $\epsilon \to 0$ as $A \to \infty$. Then we have

$$\begin{aligned}\sum_{x,n} \beta J(n)\{\cosh[a(x) - a(x+n)] - 1\}\\ &\leqslant \frac{1+\epsilon}{2} \beta A^{-2} e^{A} \big(a, (-\Delta^J) a\big)\\ &= \frac{1+\epsilon}{2} \beta A^{-2} e^{A} \left(\frac{\alpha \log K}{\beta K}\right)^2 K\\ &= \frac{\beta}{2}(1+\epsilon)(1+\delta)^{-2} K^{-1+(\alpha/\beta)(1+\delta)}\end{aligned} \tag{22}$$

which is bounded uniformly in $K \geqslant 1$ provided $\gamma \equiv \alpha/\beta \leqslant 1$. [$\delta = 0$ for $\nu = 1$ and $\delta = O(K^{-1})$ for $\nu = 2$.]

Finally

$$-[a(0) - a(\zeta)] = \frac{2\alpha}{\beta} \frac{\log K}{K} C_0(\zeta) = -\frac{\alpha}{\beta} \log 2|C_0(\zeta)|. \tag{23}$$

So it suffices to choose $\gamma = \alpha/\beta \leqslant 1$. ■

Remark 2. If supp$J$ is bounded, $M(\zeta; J)$ is bounded uniformly in ζ. This is easily seen by showing that $|C_0(x) - C_0(x+n)|$ is bounded uniformly in x if $|n| \leqslant$ const. In this case $a(x) = [C_0(x) - C_0(x - \zeta)]/\beta$ is the best choice as in the case of the Villain model.[9]

Proof of Theorem 2 (Quantum Model Cases). We restrict ourselves to the quantum Heisenberg model of spin S defined by Eq. (3):

$$\langle f \rangle = Z^{-1} \mathrm{Tr}\, f e^{\beta H}, \qquad f = \sigma_0 \sigma_\zeta = \sum_{i=1}^{3} \sigma_0^i \sigma_\zeta^i \tag{24}$$

Consider the following transformation:

$$\begin{aligned}\sigma_x^1 &\to \hat{\sigma}_x^1 = \cos\theta_x \sigma_x^1 + \sin\theta_x \sigma_x^2\\ \sigma_x^2 &\to \hat{\sigma}_x^2 = -\sin\theta_x \sigma_x^1 + \cos\theta_x \sigma_x^2\\ \sigma_x^3 &\to \hat{\sigma}_x^3 = \sigma_x^3\end{aligned} \tag{25}$$

Since this is implemented by a unitary operator on $C^{(2S+1)|\Lambda|}$, the expectation value is left invariant by this transformation. Obviously

$$Z(\theta) = \mathrm{Tr}\, e^{\beta \hat{H}}$$

is independent of $\{\theta_x;\, x \in \Lambda\}$, where $\hat{H} = H(\hat{\sigma})$. Let $\hat{f} = f(\hat{\sigma})$. Then

$$\langle f \rangle = Z^{-1} \int \left(\Pi \frac{d\theta_x}{2\pi}\right) \mathrm{Tr}\, \hat{f} e^{\beta \hat{H}} = \frac{3}{2} Z^{-1} \int \left(\Pi \frac{d\theta_x}{2\pi}\right) \mathrm{Tr}\, \hat{g} e^{\beta \hat{H}}$$

where

$$g=\sum_1^2 \sigma_0^i\sigma_\zeta^i, \qquad \hat{g}=g(\hat{\sigma})=e^{i(\theta_0-\theta_\zeta)}\hat{g}_+ + e^{-i(\theta_0-\theta_\zeta)}\hat{g}_-$$

$$\hat{g}_\pm=\frac{1}{2}\left[\sum_1^2 \sigma_0^i\sigma_\zeta^i \pm i\left(\sigma_0^1\sigma_\zeta^2-\sigma_0^2\sigma_\zeta^1\right)\right]$$

Let $\theta_x \to \theta_x \pm ia(x)$ ($+$ for $\hat{g}_+$, $-$ for $\hat{g}_-$). Then

$$\begin{aligned}
\hat{H}\to\sum_{x,n} j(n)\Bigg[&\cos(\theta_x-\theta_{x+n})\sum_1^2 \sigma_x^i\sigma_{x+n}^i \\
&-\sin(\theta_x-\theta_{x+n})\left[\sigma_x^1\sigma_{x+n}^2-\sigma_x^2\sigma_{x+n}^1\right]+\sigma_x^3\sigma_{x+n}^3\Bigg] \\
&+\sum_{x,n} j(n)\left\{\cosh\left[a(x)-a(x+n)\right]-1\right\} \\
&\times\left\{\cos(\theta_x-\theta_{x+n})\sum_1^2 \sigma_x^i\sigma_{x+n}^i-\sin(\theta_x-\theta_{x+n})\left[\sigma_x^1\sigma_{x+n}^2-\sigma_x^2\sigma_{x+n}^1\right]\right\} \\
&\pm i\sum_{x,n} j(n)\sinh\left[a(x)-a(x+n)\right] \\
&\times\left[-\sin(\theta_x-\theta_{x+n})\sum_1^2 \sigma_x^i\sigma_{x+n}^i+\cos(\theta_x-\theta_{x+n})\left(\sigma_x^1\sigma_{x+n}^2-\sigma_x^2\sigma_{x+n}^1\right)\right] \\
&\equiv \hat{H}+\delta H\pm iH'
\end{aligned}\tag{26}$$

in this order, where $\hat{H}=H(\hat{\sigma})$, δH and H' are self-adjoint operators and each pair $(x, x+n)$ is taken only once in the sum. Then

$$\begin{aligned}
&\|\exp(\beta\hat{H}+\beta\delta H\pm i\beta H')\|_1 \\
&\quad=\lim_{m\to\infty}\left\|\left[\exp(\beta\hat{H}/m)\exp(\beta\delta H/m)\exp(\pm i\beta H'/m)\right]^m\right\|_1 \\
&\quad\leqslant \|\exp\beta\hat{H}\|_1\exp\left[\beta\|\delta H\|_\infty\right]=Z\exp\left[\beta\|\delta H\|_\infty\right]
\end{aligned}$$

Thus using $|j(n)|(\cosh a-1)\leqslant J(n)(\cosh a-1)$ as before, we have

$$\begin{aligned}
|\langle f\rangle|\leqslant\tfrac{3}{2}(\|\hat{g}_+\|_\infty+\|\hat{g}_-\|_\infty)\exp\Big\{&-\left[a(0)-a(\zeta)\right]+\beta K\sum_{x,n}J(n) \\
&\times\left[\cosh(a(x)-a(x+n))-1\right]\Big\}
\end{aligned}\tag{27}$$

where each pair $(x, x+n)$ is taken only once and (since $n \neq 0$)

$$K = \left\| \sum_{1}^{2} \sigma_x^i \sigma_{x+n}^i \right\|_\infty + \|\sigma_x^1 \sigma_{x+n}^2 - \sigma_x^2 \sigma_{x+n}^1\|_\infty \leqslant \text{const}\, S^2$$

Thus the problem has been reduced to the classical XY model. ■

Remark 3. In Ref. 4, Fröhlich and Pfister remarked that the McBryan–Spencer method can be extended to quantum models, but they gave no proof.

3. REMAINING PROBLEMS AND DISCUSSION

Even though the bounds in Theorems 1 and 2 are optimal in the sense that there exists spontaneous magnetization for large β whenever the upper bounds do not decay, the decay rate in Theorem 2 may not be optimal. This is because we could not control $\cosh[a(x) - a(x+n)]$ in an optimal way which can increase rather rapidly as $|\zeta| \to \infty$ whenever $\operatorname{supp} J$ is unbounded. As a result the upper bound in Theorem 2 fails to satisfy L^1-clustering for $\nu = 1$ even if $\sum |n|\,|J(n)| < \infty$. On the other hand the upper bound for the Villain model satisfies this requirement, and then we conjecture that the optimal upper bound may be given as $\exp[\text{const}\, C_0(\zeta)/\beta]$ for any model. In fact in the one-dimensional Villain model with nearest-neighbor interaction, this upper bound gives the precise correlation functions.[5] (In fact it is easy to see that the correlation functions are given by the Gaussian integral in this case. Then the McBryan–Spencer method = approximation method by Gaussian integral[9] is precise in this case.)

After almost finishing this work, the author became aware of Ref. 7, where a similar upper bound is obtained for models which satisfy the Bogolyubov inequality. In Ref. 7 RP is not essential and not assumed, and on the other hand in our work the upper bound is improved. ($\gamma = 1/2$ in Ref. 7.) In this work reflection positivity is used to obtain $\max|C_0(x) - C_0(x-\zeta)| = |C_0(\zeta)|$. But $|C_0(x) - C_0(x-\zeta)|$ tends to zero as $|x| \to \infty$ (the Riemann–Lebesgue lemma) for any potential. Then reflection positivity may be removed from our assumptions. [If reflection positivity is not assumed, $C_0(x)$ can change rather violently especially for small $|x|$. But this will be controllable since we are interested in large $|x|$ and $|J(n)|$ must tend to zero as $|n| \to \infty$.]

The author hopes that these problems will be solved in a future publication.

Remark 4. The conjectured upper bound $\exp[\text{const}\, C_0(\zeta)/\beta]$ may hold only for large β. In fact this means a faster decay for small β, and on the other hand the decay seems to be slower than that of the long-range potential.

APPENDIX

Lemma A.1. Let $\{J(n) \geqslant 0;\ n \in Z^\nu\}$ be RP ($\nu = 1, 2, \ldots$) for the x_1 direction. Let $x = (x_1, 0, \ldots, 0)$, $x_1 = 1, 2, \ldots$. Then $C_0(x)$ is monotone decreasing and convex in $|x| = |x_l|$ ($\neq 0$).

Proof. Introducing an artificial mass $M \geqslant 0$, we consider

$$\begin{aligned} C_M(x) &= (M^2 - \Delta^J)^{-1}(x,0) \\ &= \langle q(0)q(x)\rangle \\ &= \frac{1}{Z}\int q(0)q(x)\exp\Big[-\tfrac{1}{2}\big(q,(M^2-\Delta^J)q\big)\Big]\Pi\, dq \end{aligned} \tag{A.1}$$

where $q(y) \in R$ for all $y \in Z^\nu$. Since J is RP, there exists a positive measure $d\rho(\mu)$ (defined through the transfer matrix[3]) such that

$$\langle q(0)q(x)\rangle = \int_0^\infty \exp[-\mu|x|]\, d\rho(\mu) \tag{A.2}$$

where $\exp[-\mu|x|]$ is monotone decreasing and convex in $|x|$. Then so is $C_M(x)$ and so is $C_0(x)$, too. ∎

Since the reflection positivity itself is not necessary in our theorems, it may be interesting to know to what extent our assumptions can be weakened.

Lemma A.2. Let $\{J(n) \geqslant 0;\ n \in Z^\nu,\ n \neq 0\}$ depend only on $(|n_1|, \ldots, |n_\nu|)$, and let $J(n)$ be monotone decreasing in $|n_i|$. Then $C_0(x) = C_0((|x_1|, \ldots, |x_\nu|))$ is monotone decreasing in $|x_i|$.

Proof. Let $\delta_n(x, y) = 1$ for $y = x + n$ and 0 otherwise. Noticing that $J(n) = J(-n)$, we consider

$$(M^2 - \Delta^J)^{-1} = \Big(M^2 + \sum_n J(n) - \sum_n J(n)\delta_n\Big)^{-1}$$

or equivalently

$$\Big(1 - \sum_n \hat{J}(n)\delta_n\Big)^{-1}(x,0) = \delta_{x,0} + \sum_{N=1}^{\infty} \psi^{(N)}(x) \tag{A.3}$$

where

$$\psi^{(1)}(x) = \hat{J}(x)$$
$$\psi^{(N)}(x) = \sum_n \hat{J}(n)\psi^{(N-1)}(x-n)$$

and

$$\hat{J}(n) = \frac{J(n)}{M^2 + J(0) + \sum_{m\neq 0} J(m)}, \qquad n \in Z^{\nu}$$

Then $\sum \hat{J}(n) < 1$ and $J(0)$ is some positive number bigger than $J(n)$, $|n| = 1$. [$J(0)$ can be chosen arbitrarily since $\delta_0 = 1$.] In the case $\nu = 1$,

$$\begin{aligned}\psi^{(N)}(x) - \psi^{(N)}(x+1) = \sum_{n=1}^{\infty} &\left[\hat{J}(n) - \hat{J}(n+1)\right] \\ &\times\left[\psi^{(N-1)}(x-n) - \psi^{(N-1)}(x+n-1)\right] \\ &+\left[\hat{J}(0) - \hat{J}(1)\right]\left[\psi^{(N-1)}(x) - \psi^{(N-1)}(x+1)\right]\end{aligned} \tag{A.4}$$

Since $\psi^{(N)}(-x) = \psi^{(N)}(x)$ for all N, we see that $\{\psi^{(N)}(x)\}$ are monotone decreasing in $|x| \neq 0$ by induction with respect to N. The discussion is similar for $\nu \geqslant 2$. ■

But the convexity may not follow in this way. If $\nu = 1$, any RP potential $\{J(n) = J(-n);\ n \geqslant 1\}$ is given[3] by

$$K\delta_{n,1} + \int_0^{\infty} e^{-n\mu}\left[\rho_+(\mu) + (-1)^{n-1}\rho_-(\mu)\right] d\mu \tag{A.5}$$

where $K, \rho_{\pm} \geqslant 0$. Then if $\rho_- = 0$, this is also monotone decreasing and convex in $|n|$. By setting $\rho_- = 0$, $\rho_+ \cong \mu^{\alpha-1}\log^{\alpha_1}\mu \ldots \log_s^{\alpha_s}\mu$ we find that

$$J(n) \cong n^{-\alpha}\log^{\alpha_1}n \ldots \log_s^{\alpha_s}n \qquad (\alpha > 0\colon s = 1,2,\ldots)$$

is RP. This is also RP for $\nu = 2, 3 \ldots$.

For $\nu \leqslant 2$, $J(n) \cong n^{-2\nu}$ is critical. In this case

$$\tilde{C}(k) \cong \begin{cases} \dfrac{1}{|k|}, & \nu = 1, \\ \dfrac{1}{k^2}|\log k|^{-1}, & \nu = 2 \end{cases}$$

then $-C_0(\zeta) \to \infty$ as $|\zeta| \to \infty$. This is also the case for $J(n) \cong n^{-2\nu}$ $\log n \ldots \log_s n$ (see Refs. 3, 7).

Lemma A.3. Let $\nu = 2$ and let $\{J(n) \geqslant 0;\ n \in Z^2,\ n \neq 0\}$ satisfy (A′). Then (A) holds.

Proof.

$$\tilde{C}(k)^{-1} = \sum_n J(n) 2\sin^2 \frac{kn}{2}$$

$$= \sum_{|n| \leqslant |k|^{-\delta}} \left[J(n)\frac{(kn)^2}{2} + J(n) O\big((kn)^4\big) \right]$$

$$+ \int_{|n| \geqslant |k|^{-\delta}} d^2 n\, J(|n|) 2\sin^2 \frac{kn}{2} + R(k)$$

where $J(|n|) = J((|n|, 0))$ for $|n| \in Z$ and $J(|n| + s) = (1 - s)J(|n|) + sJ(|n| + 1)$ for $|n| \in Z$ and $s \in [0, 1]$. $0 < \delta < 1$ is chosen so that

$$\sum_{|n| \leqslant |k|^{-\delta}} J(n)|n|^4|k|^4 \leqslant \text{const}\,|k|^{2+\delta_1}$$

with $\delta_1 > 0$. Since $J(n) \leqslant \text{const}\,|n|^{-3-\epsilon_1}$, $\delta = 1/3$ is sufficient. The remaining term $R(k)$ is bounded by

$$\text{const} \sum_{|n| \geqslant |k|^{-\delta}} \left[\sup_{l \in [0,1]^2} \left| J(n)\sin^2 \frac{kn}{2} - J(n+l)\sin^2 \frac{nk + kl}{2} \right| \right]$$

$$\leqslant \text{const} \sum_{|n| \geqslant |k|^{-\delta}} \left[\sup_l |J(n) - J(n+l)| n^2k^2 + J(n)|n|k^2 \right]$$

$$\leqslant \text{const} \sum_{|n| \geqslant |k|^{-\delta}} \left(|n|^{-2-\epsilon_1} + |n|^{-2-\epsilon_2}\right)k^2 \leqslant \text{const}\,|k|^{2+\delta_2}$$

for some δ_2. Let

$$F(k) = \sum_{|n| \leqslant |k|^{-\delta}} J(n)\frac{(kn)^2}{2} + \int_{|n| \geqslant |k|^{-\delta}} d^2 n\, J(|n|) 2\sin^2 \frac{kn}{2}$$

Then this is a rotationally invariant function of $k = (k_1, k_2)$. In fact $n_1 n_2 k_1 k_2$ in $(nk)^2$ vanishes by symmetry $(n_1, n_2) \to (n_1, -n_2)$. Now $\tilde{C}(k)^{-1}(k) \geqslant ck^2$, $c > 0$ for $|k| \leqslant \epsilon$. {Since J is RP for the two directions, this is the case for all $k \in [-\pi, \pi)^2$. This follows from $J((1,0)) = J((0,1)) \neq 0$ which also follows from the expression in Ref. 3. See for example (A.5) which shows this fact for $\nu = 1$.} Then

$$\tilde{C}(k) = \left\{ F(k)\left[1 + O(|k|^{\delta_3})\right] \right\}^{-1}$$

$$= \left[1 + O(|k|^{\delta_3})\right]\left[\frac{1}{F(k)} + \frac{1}{1 + O(|k|^{\delta_3})} \right]$$

where $\delta_3 > 0$, $|k| \leqslant \epsilon$ and $F(k) \leqslant O(|k|^{\delta_3})$ has been assumed without loss of generality. Since $F(k) \geqslant \text{const}\, k^2$ for $|k| \leqslant \epsilon$, $O(|k|^{\delta_3})F(k)^{-1}$ is integrable since $\nu = 2$. Then A holds since $\tilde{C}(k)^{-1} = 0$ only for $k = 0$ if $k \in [-\pi, \pi]^2$. ∎

Lemma A.4. Let $\nu = 1$ and let $\sum_1^\infty nJ(n) < \infty$ $[J(n) \geqslant 0]$. Then L^1 clustering holds for the Villain model:

$$\sum_x |\langle s_0 s_x \rangle| < \infty \tag{A.6}$$

Proof.

$$\begin{aligned}
\tilde{C}(k)^{-1} &= \sum J(n) 2 \sin^2 \frac{nk}{2} \\
&\leqslant \left[\sum |n| J(n) \left| \sin \frac{nk}{2} \right| \right] |k| \\
&= \left(\sum_{|n| \leqslant |k|^{-1/2}} + \sum_{|n| > |k|^{-1/2}} \right) |n| J(n) \left| \sin \frac{nk}{2} \right| |k| \\
&\leqslant \left[\text{const}\, |k|^{1/2} + \sum_{|n| > |k|^{-1/2}} |nJ(n)| \right] |k| \\
&= g(|k|)|k|
\end{aligned}$$

where $g(|k|)\downarrow 0$ as $|k|\downarrow 0$. Now

$$\int_{-\pi}^{\pi} \frac{e^{ikx} - 1}{|k|}\, dk = -\int_0^{\pi} 4 \sin^2 \frac{xk}{2} \frac{dk}{|k|} \cong -\text{const} \log |x|$$

Then $C_0(x) \leqslant -D(x) \log |x|$, where $D(x) \to \infty$ monotonically as $|x| \to \infty$. This means

$$\exp\left[(1/\beta) C_0(x) \right] \leqslant \text{const}(1 + |x|)^{-\alpha}$$

for any $\alpha > 1$. ∎

ACKNOWLEDGMENTS

The author would like to thank L. Landau, J. Westwater, and G. Reents for stimulating discussions and encouragement. Especially L. Landau kindly showed the author Refs. 6 and 7, which were helpful in writing this paper.

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